A GPU implementation of an asset-liability management model for insurance companies

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and
Analistas Financieros Internacionales (AFI)

Financial Mathematics and Supercomputing
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Project

This project is part of a contract between:

- International Financial Analysts, AFI

  http://www.afi.com

- Technological Institute for Industrial Mathematics, ITMATI

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Goal

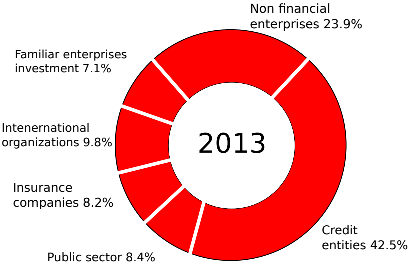
To develop and commercialize an ALM software for Life Insurance Companies

AFI

International Financial Analysts (AFI)

http://www.afi.es

- Financial consulting firm, located in Madrid
- Works for the most important banks and non financial enterprises in Spain



AFI

Shareholders:

- Strong number of shareholders: 25 shareholders
- Average time in AFI, 14 years
- Average time working in financial industry, 21 years

Staff:

- Stable team with 70 consultants and analysts
- About 50 % have a Phd or master
- Average age is 33 years
- Average experience time working in financial industry, 10 years
- Multidisciplinary team: mathematicians, informatics, economists, actuaries

AFI (some collaborations with university)

- Long term collaboration with the Master in Industrial Mathematics

  `http://www.m2i.es`

  Interuniversity master:

  Galicia: U. of Santiago, U. Vigo and U. A Coruña

  Madrid: U. Carlos III and U. Polytechnic

- Several previous contracts with M2NICA (A Coruna research group)

- Non-academic beneficiary in the

  EID-Marie Curie-H2020 WAKEUPCALL Project

  Academic: CWI-Delft (National research institute for mathematics and computer science in the Netherlands) - Bologna(Italy) - UDC(Spain)

  (2015-18 European Industrial Doctorate funded by EU)

ITMATI

Technological Institute for Industrial Mathematics, ITMATI

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Consortium created in 2011 by the following universities:

- Universidade da Coruña
- Universidade de Santiago de Compostela
- Universidade de Vigo

Located in Santiago de Compostela

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- Develop a model for the Asset-Liability Management (ALM) of a life insurance company

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and

- Carry out its efficient parallel numerical implementation using Graphics Processing Units (GPUs) hardware.

1. Asset liablity management (ALM)
2. Insurance company: asset and liability portfolios
3. Mathematical models
4. Numerical implementation
5. GPUs parallelization

Asset Liability Management (ALM)

But...What is ALM?

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- Asset: a resource with the expectation that it will provide future benefit

  Right to receive money

- Liability: debts or obligations that arise during the course of business operations.

  An obligation to pay

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For each asset you have an income (revenue/earning) $->$ cash inflow (+)

For each liability you have an expense $->$ cash outflow (-)

Asset Liability Management (ALM)

Simple example: a family assets and liabilities portfolio (domestic economy)

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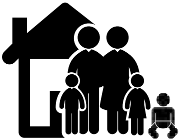

Assets	Liabilities
Salaries	House mortgage (or rental)
Some cash in the bank	A loan to buy a car
Bank deposits	Monthly bills: water, gas, electricity, telephone
Fixed interest rate assets	Taxes for the vehicle, the house...
Stocks	Food
Renting of a house	Spare time activities, holidays travels
	Weekly children assignment

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Simple example: a family assets and liabilities portfolio (domestic economy)

This family needs to predict all these things to

- avoid to having a liquidity risk (that the family is not able to face its obligations)
- smartly reinvest the surplus (the remaining money) after meeting all its obligations

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Hopefully...in this case, the familiar asset-liability management is easy, because

- Small portfolios: there is a low number of assets and liabilities
- The incomes and payments are almost fixed: they do not vary with time

Asset Liability Management (ALM)

ALM

Broad denomination for models that are used to forecast the evolution of a company along time, projecting together their assets and liabilities portfolios and computing the predicted cash inflows and outflows

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  - An insurance company,

  - Or, more generally, even any financial institution, a state pension fund or even a non financial corporation (big enterprise, with huge and diversified assets and liabilities)...

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- Depending on the business model of the company, the specific definition of the underlying models for the assets and liabilities may vary.

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- Cover liquidity and interest rate risks in order to ensure the solvency of the company, that is, its capability to meet all its financial obligations

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Maximise investment returns, while minimizing reinvestment risks

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ALM has a particular relevance in insurance industry, because the central problem in insurance business is precisely to guarantee the solvency of the company

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Static and dynamic ALM models

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- Mainly due to new regulations and a stronger competition:

  - With Solvency II, insurance companies are allowed, and even encouraged, to develop their own in house ALM models and simulators to asses their risks

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- The increase in computational power(new hardware architectures), allows the computation of more precise aproximations of the portfolio evolution using models of increasing complexity

Liability portfolio

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With-profit life insurance policy

Contract between the insurance company and the insured, known as the policyholder, where the insurer pays monthly a variable rate premium to the policyholder, and also pay a premium at maturity (or policy expiration) or if the insured dies before the policy expiry.

Usually the policyholder has the possibility of cancelling the policy before maturity (rescue the policy) paying a fee.

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- There can be hundreds of thousands of such policies in the liability portfolio
- Stand alone computation: compute the ALM projection taking into account all the policies (one by one)
- Model points computation: usually similar policies are grouped in buckets, so called model points, to reduce the computational cost of the calculus

  Example: group all the policies of policyholders between 30 and 35 years, with 10 years expiration date (data mining)

Liability portfolio

Liability portfolio

For each policy of the portfolio we know:

- Principal: n_i
- Age of the policyholder: a_i
- Maturity: T_i
- Premium: π_i
- Bonus: b_i

Asset portfolio

The asset portfolio contains fixed interest rate instruments (bonds), equities and cash

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- Bonds portfolio

N_0	N_1	N_2	$\dots$	N_{n-1}	N_n
c_0	c_1	c_2	$\dots$	c_{n-1}	c_n

N_i: Nominal/principal i

c_i: Coupon i

n: maximum maturity of bonds

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n: maximum maturity of bonds

- Equity. The remaining is invested in equity (variable rates instruments)
- Cash (current account)

Balance sheet

- In financial accounting, a balance sheet is a summary of the financial balances of a corporation or any business organization
- A balance sheet can be understood as a "snapshot of a company's financial condition", taking into account the values of assets and liabilities at a particular closing date t
- There is a bunch of usual/standard calculus that a company computes to understand its financial position at a certain date

Balance sheet

Assets	Liabilities
Value of bonds	Maturity Payment
Value of equity	Death Payment
Value of cash	Surrender Payment
Financial incomes	Policy Payment
Asset duration	Liability fluxes
P&L	Reserves
	Liabilities duration

Balance sheet -> Liabilities computations

Policy benefits, $M_k + D_k + S_k$

- **Maturity payments,** M_k
  Payments that the company must make for all those policies that reach maturity at time k:

$$M_k = \sum_{m_i \in \mathscr{P}} a_k n^a_{i,k-1} \,,$$

  $n^a_{i,k-1}$: nominal of the policies in m_i with maturity at month k, i.e. $t_k = T_i$
  a_k: the discount factor at month k

- **Death payments,** D_k, the payments that the company must make for all those policies whose policyholders die at time k:

$$D_k = \sum_{m_i \in \mathscr{P}} n^d_{i,k} a_k p_d \,,$$

  p_d: the death premium (for example $p_d = 105\,\%$).
  The nominal associated to dead policyholders depends on a biometric model

Balance sheet -> Liabilities computations

Policy benefits, $M_k + D_k + S_k$

- **Surrender payments,** S_k
  Payments that the company must make for all those policyholders who surrender (recues the policy before maturity) before maturity at period k:

$$S_k = \sum_{m_i \in \mathscr{P}} s_k^i,$$

s_k^i: payment if the policyholder surrenders at model point m_i. Moreover, we have:

$$s_k^i = n_{i,k}^s a_k p_s \,,$$

where p_s is the guaranteed payment if the policyholder surrenders

Balance sheet -> Liability computations

Supposed/estimated/approximated liability cash flows

- These are the supposed liabilities cash flows, without taking into account the possible surrenders. They are an estimation of the real liability cashflows.
- Supposed liability (due to death or maturity) cash flow for each model point m_i at time t_k

$$l_i^F(t_k) = \begin{cases} n_{i,k}^d p_d & \text{if } t_k < T_i \,, \\ n_{i,k}^a & \text{if } t_k = T_i \,, \\ 0 & \text{otherwise.} \end{cases}$$

- We say supposed, because they are fixed (because the death numbers are given by the company life tables and the maturities are known), and we are not computing the stochastic part of the payments that are given by the surrenders. They do not depend on the simulation (scenario).

Supposed liability cash flows

L_k^F, are the sum of the supposed future cash flows for each model point:

$$L_k^F = \sum_{m_i \in \mathscr{P}} a_k l_i^F(t_k) \cdot n_k^i \,.$$

Balance sheet -> Liability computations

- **Reserves**

The actuarial reserves represent the required money at a given time instant tk to face the future obligations. The actuarial reserve t_k for the model point m_i, v_k^i, is given by

$$v_k^i = \sum_{j=k+1}^{M} a_j l_i^F(t_j),$$

with $l_i^F(t_j), j = k+1, \ldots, M$:

$$l_i^F(t_j) = \begin{cases} n_{i,j}^d d_j^i p_d & \text{if } t_j < T_i \\ n_{i,j}^a l_j^i & \text{if } t_j = T_i \\ 0 & \text{otherwise.} \end{cases}$$

l_j^i the percentage of policies that stay alive from the $j-1$ to the j–th period.

- **Total reserves and variation of reserves** at time t_k

$$V_k = \sum_{m_i \in \mathscr{P}} v_k^i, \quad \Delta V_k = V_k - V_{k-1}, V_0 = 0$$

Balance sheet -> Liability computations

- **Duration of liabilities**

We use the Macaulay duration measure:

$$L_k^D = \frac{\frac{1}{12} \sum_{i=1}^{\max(n,M-k)} i \cdot d_i \cdot L_{k-1+i}^F}{\sum_{i=1}^{\max(n,M-k)} d_i \cdot L_{k-1+i}^F},$$

It is a measure of interest rate sensitivity to match assets and liabilities.

Balance sheet -> Asset computations

Value of the asset portfolio, A_k^ν:

$$A_k^\nu = B_k^\nu + E_k^\nu + C_k^\nu .$$

where:

- B_k^ν, Value of bonds portfolio
- E_k^ν, Value of equity
- C_k^ν, Value of cash

Balance sheet -> Asset computations

- **Value of bonds portfolio**

$$B_k^v = \bar{N}_k + \bar{\bar{c}}_k \,,$$

where $\bar{N}_k$ is the value of the nominals/principals

$$\bar{N}_k = \sum_{i=1}^{n} d_i \cdot N_i \,,$$

and $\bar{\bar{c}}_k$ is the value of the coupons

$$\bar{\bar{c}}_k = \frac{1}{12} \sum_{i=1}^{n} \hat{d}_i \cdot c_i \cdot N_i \,.$$

- **Value of equity**: can be obtained recursively

$$E_k^v = E_{k-1}^v \cdot (1 + E_k^p)$$

where the equity profitability:

$$E_k^p = \frac{\hat{S}_k}{\hat{S}_{k-1}} - 1 \,,$$

Balance sheet -> Asset computations

- **Financial incomes (revenue, profits or turnover) and value of the cash**
  Revenue or turnover: income that a company receives from its normal business activities
  Financial revenues at month k, I_k, incomes that come from the bonds investment

$$I_k = C_k + A_k \,,$$

  where:

  - C_k denotes the revenues associated to coupon payments at month k

$$C_k = \sum_{i=1}^{n} N_i c_i .$$

  - A_k are the *amortizations*, that is the nominal whose maturity is in the current month k, which is stored in position zero of the bonds table

$$A_k = N_0 \,.$$

- **Duration of the assets,** A_k^D, we use the Macaulay duration

$$A_k^D = \frac{\frac{1}{12}\left[\sum_{i=0}^{n} i \cdot N_i \cdot d_i + \sum_{i=0}^{n} i \cdot c_i \cdot \hat{d}_i\right]}{B_k^v} \,,$$

Balance sheet -> Asset computations

- **Profit and Loss (P&L)**

The profit and loss account (P&L) is a primary financial statement that summarizes the revenues, costs and expenses incurred during a specified period of time.
The P&L at month k, PL_k, is given by:

$$PL_k = \Delta V_k + I_k - (M_k + D_k + S_k),$$

so that the accumulated value of the P&L at moth k, A_k^{PL}, is given by:

$$A_k^{PL} = A_{k-1}^{PL} + PL_k,$$

starting from $A_0^{PL} = 0$.

Mathematical models

Our ALM model consists of two coupled models: one for the assets portfolio and one for the liabilities portfolio.

Assets and liabilities evolve together in time, interacting each other. The asset and liability models are coupled by the asset allocation model

- Liability model**S**
- Asset model**S**
- Asset allocation model, that couples both models

Mathematical models -> Asset model

We have stochastic models for

- Interest rates evolution
- Stock price evolution

Asset models -> Bonds interest rates model

Continous interest rates model

Black and Karasinski (BK) model for the interest rates

Assume that the short rate $r(t)$ follows a lognormal process under the risk-neutral measure Q. Thus, the logarithm of the instantaneous spot rate, $\ln r(t)$, evolves according to

$$d \ln r(t) = [\theta(t) - \alpha(t) \ln(r(t))]dt + \sigma(t)dW(t),$$

where $r(0) = r_0$ and

- $\theta(t)$, mean reversion level
- $\alpha(t)$, mean reversion speed
- $\sigma(t)$ instantaneous volatility of $\log r(t)$
- $W(t)$, standard Brownian motion under the risk-neutral measure Q.

Asset models -> Bonds interest rates model

Considering the long term rate $\mu(t)$, we can rewrite:

$$d\ln r(t) = \alpha(t)[\ln\mu(t) - \ln r(t)]dt + \sigma(t)dW(t),$$

In our case we will use constant parameters, α, σ and μ.

Discrete interest rate model

In order to simulate the BK model we use the Euler-Maruyama scheme

$$\ln r(t_{k+1}) = \ln r(t_k) + \alpha \cdot [\mu - \ln r(t_k)]\,\Delta_k + \sigma\sqrt{\Delta_k}\,(\rho \cdot Y(t_k) + \sqrt{1-\rho^2}\cdot Z(t_k)).$$

And taking exponentials

$$r(t_{k+1}) = \exp\left\{\ln r(t_k) + \alpha\,[\mu - \ln r(t_k)]\,\Delta_k + \sigma\sqrt{\Delta_k}\,(\rho \cdot Y(t_k) + \sqrt{1-\rho^2}\cdot Z(t_k))\right\},$$

where ρ denotes a correlation and Y and Z are independent $\mathcal{N}(0,1)$ distributed.

Asset models -> Stock price model

Continous stock price model

Assume that the stock price $S(t)$ follows a Geometric Brownian Motion

$$d\hat{S}(t) = \hat{\mu}\hat{S}(t)dt + \hat{\sigma}\hat{S}(t)d\hat{W}(t),$$

where

- $\hat{\sigma} > 0$ stock volatility
- $\hat{\mu} \in \mathbb{R}$ stock drift

and again $\hat{W}(t)$ denotes a standard Brownian motion.

Discrete stock price model

Log-Euler discretization of the stock model:

$$\hat{S}(t_k) = \hat{S}(t_{k-1}) \cdot \exp\left\{\hat{\mu} \cdot \Delta_k + \sqrt{\Delta_i} \cdot \hat{\sigma} \cdot (\hat{\rho} \cdot Y + \sqrt{1 - \hat{\rho}^2} \cdot \hat{Z})\right\},$$

where $\hat{\rho}$ denotes a correlation and Y and $\hat{Z}$ are independent $\mathcal{N}(0, 1)$ distributed.

Asset models -> Dynamic asset allocation model

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- This investment strategy can be parameterized and is an input for the ALM model. So, at each date we must revbalance (restructure) the portfolio to ensure that there is enough money to pay to the company's policyholders and the policies shareholders, and also maximize the income of the company

Asset models -> Dynamic asset allocation model

Steps of the dynamic asset allocation model

- Evaluation of bond position with respect to target asset portfolio
- Computation of the gap between the durations of assets and liabilities. The duration gap is a measure of the risk due to changes in the interest rate. This measures the a mismatch between asset and liability.
- Decision of reinvestment strategy

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- **Biometric model**

  For the premium that the company must pay in case of death of the policyholder before the the policy's maturity.

Liability models -> Surrender model

- To compute the percentage of policy owners that rescue their policy, we compare the current interest rates with the profit offered by the insurance company
  Δr_k, gap between averaged interest rate obtained from the BK model and the technical rate:
  $$\Delta r_k = max(A_k^p - \bar{r}_k, 0),$$
  $barr_k$: an averaged interest rate obtained from the BK model
  A_k^p: the annualized profitability of the asset portfolio in the previous month
  $$A_k^p = 12 \cdot \Big(\frac{\hat{A}_k^v + P_{k-1}}{A_{k-1}^v} - 1\Big).$$
  $\hat{A}_k^v$: asset value, before the dynamic asset allocation

Liability models -> Surrender model

		Maturities threshold		
		I_1^m	I_2^m	I_3^m
Threshold	I_1^t	q_{11}^s	q_{12}^s	q_{13}^s
	I_2^t	q_{21}^s	q_{22}^s	q_{23}^s
	I_3^t	q_{31}^s	q_{32}^s	q_{33}^s

- In this Table I_i^t is the threshold interval i, q_{ij}^s is the non-surrender rate of the interval i for the maturity interval j. In our example, $I_1^t = [0, 3\,\%]$, $I_2^t = [3\,\%, 9\,\%]$, $I_3^t = [9\,\%, +\infty]$.
- Therefore, if the difference between the profitability of our portfolio and the average interest rate is in the threshold interval I_i^t, then the percentage of non-surrender associated to maturity interval I_j^m, is q_{ij}^s.

Liability models -> Biometric model (life tables)

The biometric model is required to compute the premium that the company must pay in case of death of the policyholder before maturity
Given the biometric model, we can obtain the life tables

month/age	a_1	a_2	a_3	...
k	$l_k^{a_1}$	$l_k^{a_2}$	$l_k^{a_3}$	...
$k+1$	$(1-q^{a_1})l_k^{a_1}$	$(1-q^{a_2})l_k^{a_2}$	$(1-q^{a_3})l_k^{a_3}$	...

$l_k^{a_i}$ denotes the probabilitiy of survival from month k to $k+1$, that can be compute using the following recursion formula

$$l_{k+1}^{a_i} = l_k^{a_i} \cdot (1-q^{a_i})$$

where q^{a_i} is a constant death rate depending on the bucket a_i and $l_0^{a_i} = 100\,\%$.
Conversly, the death table can be computed from the life table as follows

$$d_k^{a_i} = l_k^{a_i} - l_{k+1}^{a_i}$$

Note that in our case q^{a_i} are constant for each age, but we could also use an stochastic model for them

Numerical method

Balance sheet projection

- The balance sheet, and thus the value, of the company must be computed for any given economic scenario given the previously defined models
- A economic scenario is given by the evolution of several economic variables, like for example fixed or variable interest rates
- If we know all these values at any time in the scenario, we can supply these values to the assets and liabilities models, to compute the anticipated surrendered policies. Afterwards we compute the the cash inflows and outflows and the value of the assets and liabilities portfolios

Numerical method

The numerical method consists of a Monte Carlo Balance Sheet projection

We generate thousands of economic scenarios characterized by the evolution of stochastic fixed interest rate bonds or stocks. These scenarios are defined in the time interval $[T_0, T]$, and we divide this interval in N monthly spaced subintervals

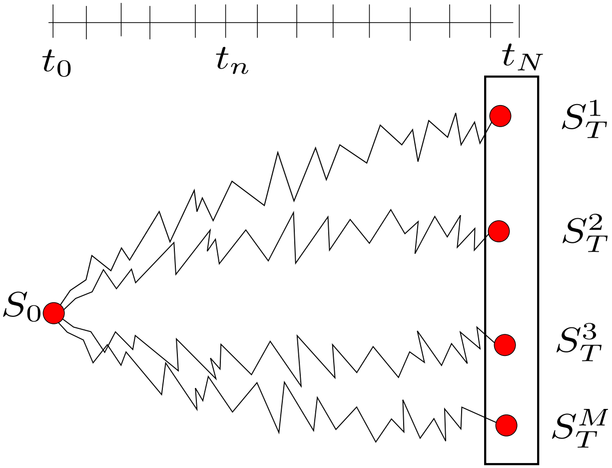
$$\{[t_k, t_{k+1}], k = 0, \ldots, N-1\}$$

with $t_0 = T_0$ and $t_N = T$

- Let's suppose that we already know all the position of the portfolios at the beginning of month t_k of a given scenario
- Then, to advance from month t_k to month t_{k+1}, firstly all the interest rates are computed following stochastic BK interest rates model.
- Then we compute the balance sheet of the company, and afterwards we apply our dynamic asset allocation model
- Advance to the next time

Numerical method

Monte Carlo balance sheet projection



Balance sheet projection

Market value	Before rebalance	Portfolio value Bonds value Cash value Equity value
	After rebalance	Portfolio value Bonds value Cash value Equity value
Liability value		
Cash flows	Bonds	Buy/sell bonds Coupons Amortizations
	Equity	Buy/sell equities
	Liabilities	Maturity payments Death payments Surrender payments
Profit and loss (P&L)		Reserves variation Variation of market value of bonds Variation of market value of equity Financial incomes Benefits

1. Asset liablity management (ALM)
2. Insurance company: asset and liablity portfolios
3. Mathematical models
4. Numerical implementation
5. GPUs parallelization

Parallel GPU implementation

- Computing the balance sheet of the company at all the times for each scenario can be a quite demanding computational task, depending on:
  - the number of policies
  - the number of scenarios
  - the temporal lenght of the forecast
  - the number of time steps per each scenario
- Therefore, the model has been parallelized in multi CPU clusters using OpenMP and also using GPUs
- The current GPU implementation has been developed for Nvidia GPUs using CUDA

GPUs

Computing power is measured in FLOPS

FLOP

Number of floating point operations per second

GPUs

Computing power is measured in FLOPS

FLOP

Number of floating point operations per second

- MEGAFLOPS (MFLOPS, 10^6 FLOPS): ONE MILLION
- GIGAFLOPS (GFLOPS, 10^9 FLOPS): THOUSAND MILLION
- TERAFLOPS (TFLOPS, 10^{12} FLOPS): 10^6 MILLION
- PETAFLOP (PFLOPS, 10^{15} FLOPS)
- EXAFLOPS (EFLOPS, 10^{18} FLOPS)

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A current GPU, Nvidia Titan Black, for example, can deliver:

- 5.1 TeraFlops in single precission
- **1.3 TeraFlops in double precision!!**

GPUs

Let's compare!

Evolution of computing power in the last 40 years

1970	STAR-100	100	MFLOPS
1972	ILLIAC IV, USA	$100-150$	MFLOPS
1976	Cray-1, USA	240	MFLOPS
1988	Cray Y-MP	1	GFLOP
1995	Cray T3E-1200E, USA	1	TFLOP
2001	Earth Simulator, Japón	35	TFLOPS
2005	BlueGene, USA	36,01	TFLOPS
2005	Cray X1, USA	50	TFLOPS
2005	MareNostrum, Spain	94,21	TFLOPS
2008	Cray XT5 (JAGUAR)	831,7	TFLOPS
2009	RoadRunner	1	PFLOPS
2010	Tianhe-1, China	$4,7$	PFLOPS
2013	Tianhe-2, China	$17,59$	PFLOPS

Less than 40 GPUs have the same power of the Cray X1, in 2001!!

And we are in the way to reaching the EXAFLOP (A TRILLION FLOATING POINT CALCULATIONS PER SECOND!)...

GPUs

Cray X1 (2005, 15 million dolars, 166,4 Tons)
64 racks of 2,600 Kilos each

Remember: less than 40 GPUs have the same double precission computing power!!
...and they fit on one small rack!

Parallel GPU implementation

Some implementation details:

- Whole code build from scratch
- Parallelized both in MultiCPU and GPU
- About 20,000 lines of C++ CUDA code
- Multiplatform library. Dinamic library available for Unix/Linux and Windows (Windows dll, Dynamic-link library)
- Full object oriented, both in the sides of the CPU and in the GPU code
- Avoiding repeated code when possible: extensive use of host-device functions Shared code between CPU and GPU
- Extensive use of Thrust
- "On the fly" random number generation. Currently using Curand (L'Ecuyer's MRG32k3a)

Numerical results

Configuration

- Model points: 10000
- Number of scenarios: 100000
- Time steps: 720

Test platform (hardware)

- Dual Quad-Core Xeon E5530 server
- 16 GB of RAM
- Nvidia GTX 580, 1.5 GB of RAM
- Cluster of the M2NICA UDC Research Group

Numerical results

Performance results

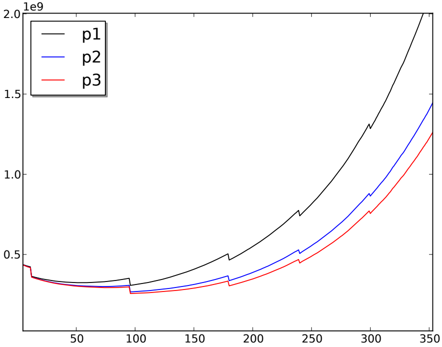
Times for MultiCPU and GPU implementations

$$\text{Speedup}= \frac{\text{time 1 CPU}}{\text{time}}$$

	1 CPU	2 CPU	4 CPU	8 CPU	GPU
Time	1061m41s	611m50s	271m40s	145m10s	40m36s
Speedup	1	1.75	3.91	7.33	26.51

Numerical results

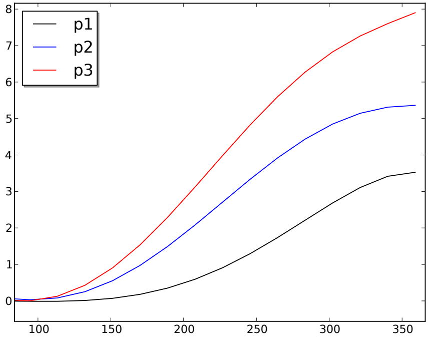
Asset portfolio value



- p_1: with death payments
- p_2: with death payments, surrender option and 10
- p_3: with death payments, surrender option and no surrender fee

Numerical results

Asset portfolio value



- p_1: with death payments
- p_2: with death payments, surrender option and 10
- p_3: with death payments, surrender option and no surrender fee

Conclussions

Tough competition:

- Two alternatives in the market: **MoSes**, **Prophet**

  \- **MoSes** (developed by Tillinghast), MoSes provides a single common platform for many financial models
  \- **Prophet**

- Very expensive
- Powerfull, complex and huge pieces of software
- They translate the business model of the company into object oriented C++ code (classes and basic methods).
- Users must then customize these C++ skeletons